On subgroups in division rings of type 2

Bui Xuan Hai*, Trinh Thanh Deo†, and Mai Hoang Bien‡

Abstract

Let D be a division ring with the center F. We say that D is a *division ring of type* 2 if for every two elements $x, y \in D$, the division subring $F(x, y)$ is a finite dimensional vector space over F. In this paper we investigate multiplicative subgroups in such a ring.

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*Faculty of Mathematics and Computer Science, University of Science, VNU-HCM, 227 Nguyen Van Cu Str., Dist. 5, HCM-City, Vietnam, e-mail: bxhai@hcmus.edu.vn

†Faculty of Mathematics and Computer Science, University of Science, VNU-HCM, 227 Nguyen Van Cu Str., Dist. 5, HCM-City, Vietnam, e-mail: ttdeo@hcmus.edu.vn

‡Department of Basic Sciences, University of Architecture, 196 Pasteur Str., Dist. 1, HCM-City, Vietnam, e-mail: maihoangbien012@yahoo.com

1 Introduction

In the theory of division rings, one of the interesting problems is the question of what groups can not occur as multiplicative groups of non-commutative division rings. There are some very interesting results answering to this question. Among them we note the famous discovery of Wedderburn in 1905 , which states that *if D^* is a finite group, then D is commutative*, where D^* denotes the multiplicative group of D. Later, L. K. Hua (see, for example, in [8, p. 223]) proved that the multiplicative group of a non-commutative division ring cannot be solvable. Recently, in [7] it was shown that the group D^* can not be even locally nilpotent. Note also Kaplansky's Theorem (see [8,(15.15), p. 259]) which states that if the group D^*/F^* is torsion, then D is commutative. There are another results of such a kind that can be found for example, in [1]- [3], [5]- [7],...

In this paper we consider this question for division rings of type 2. Recall that a division ring D with the center F is said to be *division ring of type* 2 if for every two elements $x, y \in D$, the division subring $F(x, y)$ is a finite dimensional vector space over F.

Throughout this paper the following notations will be used consistently: D denotes a division ring with the center F and D^* is the multiplicative group of D. If S is a nonempty subset of D, then we say that S is *algebraic over* F if every element of S is algebraic over F. We denote also by $F[S]$ and $F(S)$ the subring and the division subring of D generated by S over F, respectively. The symbol D' is used to denote the derived group $[D^*, D^*]$. We say that a division ring D is *centrally finite* if it is a finite dimensional vector space over F. An element $x \in D$ is said to be *radical* over a subring K of D if there exists some positive integer $n(x)$ depending on x such that $x^{n(x)} \in K$. A nonempty subset S of D is *radical* over K if every element of S is radical over K. We denote by $N_{D/F}$ and $RN_{D/F}$ the norm and the reduced norm respectively. Finally, if G is any group then we always use the symbol $Z(G)$ to denote the center of G.

2 Finitely generated subgroups

The main purpose in this section is to prove that in a non-commutative division ring D of type 2 with the center F there are no finitely generated subgroups containing F^*.

Lemma 2.1 *Let D be a division ring with the center F, D_1 be a division subring of D containing F. Suppose that D_1 is a finite dimensional vector space over F and $a \in D_1$. Then, $N_{D_1/F}(a)$ is periodic if and only if $N_{F(a)/F}(a)$ is periodic.*

Proof. Let $F_1 = Z(D_1) \supset F$, $m^2 = [D_1 : F_1]$ and $n = [F_1(a) : F_1]$. By [4, Lem. 3, p.145 and Cor. 4, p. 150], we have

$$N_{D_1/F_1}(a) = [RN_{D_1/F_1}(a)]^m = [N_{F_1(a)/F_1}(a)]^{m^2/n}.$$

Now, using the Tower formulae for the norm, from the equality above we get

$$N_{D_1/F}(a) = [N_{F_1(a)/F}(a)]^{m^2/n}.$$

Since $a \in F(a), N_{F_1(a)/F(a)}(a) = a^k$, where $k = [F_1(a) : F(a)]$. Therefore

$$N_{F(a)/F}(a^k) = N_{F(a)/F}(N_{F_1(a)/F(a)}(a)) = N_{F_1(a)/F}(a).$$

It follows that $N_{D_1/F}(a) = [N_{F(a)/F}(a)]^{km^2/n}$, and the conclusion is now obvious. ■

The following proposition is useful. In particular, it is needed to prove the next theorem.

Proposition 2.1 *Let D be a division ring with the center F. If N is a subnormal subgroup of D^* then $Z(N) = N \cap F^*$.*

Proof. If N is contained in F^* then there is nothing to prove. Thus, suppose that N is non-central. By [10, 14.4.2, p. 439], $C_D(N) = F$. Hence $Z(N) \subseteq N \cap F^*$. Since the inclusion $N \cap F^* \subseteq Z(N)$ is obvious, $Z(N) = N \cap F^*$. ■

Theorem 2.1 *Let D be a division ring of type 2. Then $Z(D')$ is a torsion group.*

Proof. By Proposition 2.1, $Z(D') = D' \cap F^*$. Any element $a \in Z(D')$ can be written in the form $a = c_1c_2 \dots c_r$, where $c_i = [x_i, y_i]$ with $x_i, y_i \in D^*$ for $i \in \{1, \dots, r\}$. Put $D_1 = D_2 := F(c_1, c_2)$, $D_3 := F(c_1c_2, c_3)$, $\dots$, $D_r := F(c_1...c_{r-1}, c_r)$ and $F_i = Z(D_i)$ for $i \in \{1, \dots, r\}$. Since D is of type 2, $[D_i : F] < \infty$.

Since $N_{F(x_i,y_i)/F}(c_i) = 1$, by Lemma 2.1, $N_{F(c_i)/F}(c_i)$ is periodic. Again by Lemma 2.1, $N_{D_i/F}(c_i)$ is periodic. Therefore, there exists some positive integer n_i such that $N_{D_i/F}(c_i^{n_i}) = 1$. Recall that $D_2 = D_1$. Hence we get

$$N_{D_2/F}(c_1c_2)^m = N_{D_2/F}(c_1)^m N_{D_2/F}(c_2)^m = 1,$$

where $m = n_1n_2$. Again by Lemma 2.1, $N_{F(c_1c_2)/F}(c_1c_2)$ is periodic; hence $N_{D_3/F}(c_1c_2)$ is periodic. By induction, $N_{D_r/F}(c_1...c_{r-1})$ is periodic. Suppose that $N_{D_r/F}(c_1...c_{r-1})^n = 1$. Then

$$N_{D_r/F}(a^n) = N_{D_r/F}(c_1...c_{r-1})^n N_{D_r/F}(c_r)^n = 1.$$

Hence, $a^{n[D_r:F]} = 1$. Therefore, a is periodic. Thus $Z(D')$ is torsion. ■

From the discussion before Corollary 8 in [9], we can obtain the following result as a corollary of the theorem above.

Corollary 2.1 *Let D be a non-commutative ring of type 2 with the center F. Then $D' \setminus Z(D')$ contains no elements purely inseparable over F.*

In [2, Theorem 1], it was proved that if D is a centrally finite division ring and D^* is finitely generated, then D is commutative. Here, in the first, we note that if D^* is finitely generated then D is even a finite field. Further, we shall prove that in a division ring D of type 2 with the center F, there are no finitely generated subgroups containing F^*. Consequently, if D is of type 2 and D^* is finitely generated then D is a finite field.

Lemma 2.2 *Let K be a field. If the multiplicative group K^* of K is finitely generated, then K is finite.*

Proof. If $char(K) = 0$, then K contains the subfield $\mathbb{Q}$ of rational numbers. Since K^* is finitely generated, in view of [10, 5.5.8, p. 113], $\mathbb{Q}^*$ is finitely generated, that contradicts to the well-known property of the group $\mathbb{Q}^*$. Thus, we have $char(K) = p > 0$. Suppose that $K^* = \langle a_1, a_2, \dots, a_r\rangle$. Then, $K = \mathbb{F}_p(a_1, a_2, \dots, a_r)$, where $\mathbb{F}_p$ is the prime subfield of K. We shall prove that a_i is algebraic over $\mathbb{F}_p$ for every $i \in \{1, 2, \dots, r\}$. Clearly, if this will be done then K will be finite. Suppose that $a = a_i$ is transcendental over $\mathbb{F}_p$ for some i. Since the subgroup $\mathbb{F}_p(a)^*$ is finitely generated, it can be written in the form

$$\mathbb{F}_p(a)^* = \Big\langle \frac{f_1(a)}{g_1(a)}, \frac{f_2(a)}{g_2(a)}, \dots, \frac{f_n(a)}{g_n(a)} \Big\rangle,$$

where $f_i(X), g_i(X) \in \mathbb{F}_p[X], g_i(a) \neq 0$ and $(f_i(X), g_i(X)) = 1$. Take some positive integer m such that

$$m > \max \big\{ \deg(f_i), \deg(g_i) | \ i \in \{1, 2, \dots, n\} \big\}$$

and an irreducible polynomial $f(X) \in \mathbb{F}_p[X]$ of degree m (such a polynomial always exists). Then, we have

$$f(a) = \Big(\frac{f_1(a)}{g_1(a)}\Big)^{m_1} \Big(\frac{f_2(a)}{g_2(a)}\Big)^{m_2} \cdots \Big(\frac{f_n(a)}{g_n(a)}\Big)^{m_n},$$

with $m_1, m_2, \dots, m_n \in \mathbb{Z}$. Since a is transcendental, $\mathbb{F}_p[a] \simeq \mathbb{F}_p[X]$, so from the last equality it follows that there exists some $i \in \{1, 2, \dots, n\}$ such that $f(X)$ divides either $f_i(X)$ or $g_i(X)$. But this is impossible by the choice of degree m of $f(X)$. Thus, we have

proved that a_i is algebraic over $\mathbb{F}_p$ for any $i \in \{1, 2, \ldots, n\}$. Therefore, K is a finite field. $\blacksquare$

Now we can prove the following theorem, which shows that in a non-commutative division ring D of type 2 there are no finitely generated subgroups of D^*, containing F^*.

Theorem 2.2 *Let D be a non-commutative division ring of type 2 with center F and suppose that N is a subgroup of D^* containing F^*. Then N is not finitely generated.*

Proof. Suppose that there is a finitely generated subgroup $N = \langle x_1, \ldots, x_n \rangle$ of D^* containing F^*. Then, in virtue of [[10], 5.5.8, p. 113], F^*N'/N' is a finitely generated abelian group, where N' denotes the derived subgroup of N.

Case 1: char$(D) = 0$.

Then, F contains the field $\mathbb{Q}$ of rational numbers and it follows that $\mathbb{Q}^*/(\mathbb{Q}^* \cap N') \simeq \mathbb{Q}^*N'/N'$. Since F^*N'/N' is finitely generated, $\mathbb{Q}^*N'/N'$ is finitely generated and consequently $\mathbb{Q}^*/(\mathbb{Q}^* \cap N')$ is finitely generated. Consider an arbitrary element $a \in \mathbb{Q}^* \cap N'$. Then $a \in F^* \cap D' = Z(D')$. By Theorem 2.1, a is periodic. Since $a \in \mathbb{Q}$, we get $a = \pm 1$. Thus, $\mathbb{Q}^* \cap N'$ is finite. Since $\mathbb{Q}^*/(\mathbb{Q}^* \cap N')$ is finitely generated, $\mathbb{Q}^*$ is finitely generated, that is impossible.

Case 2: char$(D) = p > 0$.

Denote by $\mathbb{F}_p$ the prime subfield of F, we shall prove that F is algebraic over $\mathbb{F}_p$. In fact, suppose that $u \in F$ and u is transcendental over $\mathbb{F}_p$. Then, the group $\mathbb{F}_p(u)^*/(\mathbb{F}_p(u)^* \cap N')$ considered as a subgroup of F^*N'/N' is finitely generated. Consider an arbitrary element $f(u)/g(u) \in \mathbb{F}_p(u)^* \cap N'$, where $f(X), g(X) \in \mathbb{F}_p[X]$, $((f(X), g(X)) = 1$ and $g(u) \neq 0$. As above, we have $f(u)^s/g(u)^s = 1$ for some positive integer s. Since u is transcendental over $\mathbb{F}_p$, it follows that $f(u)/g(u) \in \mathbb{F}_p$. Therefore, $\mathbb{F}_p(u)^* \cap N'$ is finite and consequently, $\mathbb{F}_p(u)^*$ is finitely generated. But, in view of Lemma 2.2, $\mathbb{F}_p(u)$ is finite, that is a contradiction. Hence F is algebraic over $\mathbb{F}_p$ and it follows that D is algebraic over $\mathbb{F}_p$. Now, in virtue of Jacobson's Theorem [8, (13.11), p. 219], D is commutative, that is a contradiction. $\blacksquare$

From Theorem 2.1 and Lemma 2.2 we get the following result, which generalizes Theorem 1 in [2]:

Corollary 2.2 *Let D be a division ring of type 2. If the multiplicative group D^* is finitely generated, then D is a finite field.*

If M is a maximal finitely generated subgroup of D^*, then D^* is finitely generated. So, the next result follows immediately from Corollary 2.2.

Corollary 2.3 *Assume that D is a division ring of type 2. If the multiplicative group D^* has a maximal finitely generated subgroup, then D is a finite field.*

By the same way as in the proof of Theorem 2.1, we obtain the following corollary.

Corollary 2.4 *Assume that D is a division ring of type 2 with the center F and S is a subgroup of D^*. If $N = SF^*$, then N/N' is not finitely generated.*

Proof. Suppose that N/N' is finitely generated. Since $N' = S'$ and $F^*/(F^* \cap S') \simeq S'F^*/S'$, it follows that $F^*/(F^* \cap S')$ is a finitely generated abelian group. Now, by the same way as in the proof of Theorem 2.1, we can conclude that D is commutative. ■

The following result follows immediately from Corollary 2.4.

Corollary 2.5 *Assume that D is a division ring of type 2. Then, D^*/D' is not finitely generated.*

3 The radicality of subgroups

In this section we study subgroups of D^*, that are radical over some subring of D. To prove the next theorem we need the following useful property of division rings of type 2.

Lemma 3.1 *Let D be a division ring of type 2 with the center F and N be a subnormal subgroup of D^*. If for every elements $x, y \in N$, there exists some positive integer n_{xy} such that $x^{n_{xy}}y = yx^{n_{xy}}$, then $N \subseteq F$.*

Proof. Since N is subnormal in D^*, there exists the following series of subgroups

$$N = N_1 \lhd N_2 \lhd \ldots \lhd N_r = D^*.$$

Suppose that $x, y \in N$ and $K := F(x, y)$. By putting $M_i = K \cap N_i, \forall i \in \{1, \ldots, r\}$ we obtain the following series of subgroups

$$M_1 \lhd M_2 \lhd \ldots \lhd M_r = K^*.$$

For any $a \in M_1 \leq N_1 = N$, suppose that n_{ax} and n_{ay} are positive integers such that

$$a^{n_{ax}}x = xa^{n_{ax}} \text{ and } a^{n_{ay}}y = ya^{n_{ay}}.$$

Then, for $n := n_{ax}n_{ay}$ we have

$$a^n = (a^{n_{ax}})^{n_{ay}} = (xa^{n_{ax}}x^{-1})^{n_{ay}} = xa^{n_{ax}n_{ay}}x^{-1} = xa^nx^{-1},$$

and

$$a^n = (a^{n_{ay}})^{n_{ax}} = (ya^{n_{ay}}y^{-1})^{n_{ax}} = ya^{n_{ay}n_{ay}}y^{-1} = ya^ny^{-1}.$$

Therefore $a^n \in Z(K)$. Hence M_1 is radical over $Z(K)$. By [5, Theorem 1], $M_1 \subseteq Z(K)$. In particular, x and y commute with each other. Consequently, N is abelian group. By [10, 14.4.4, p. 440], $N \subseteq F$. ■

Theorem 3.1 *Let D be a division ring of type 2 with the center F, K be a proper division subring of D and suppose that N is a normal subgroup of D^*. If N is radical over K, then $N \subseteq F$.*

Proof. Suppose that N is not contained in the center F. If $N \setminus K = \emptyset$, then $N \subseteq K$. By [10, p. 433], either $K \subseteq F$ or $K = D$. Since $K \neq D$ by the supposition, it follows that $K \subseteq F$. Hence $N \subseteq F$, that contradicts to the supposition. Thus, we have $N \setminus K \neq \emptyset$.

Now, to complete the proof of our theorem we shall show that the elements of N satisfy the requirements of Lemma 3.1. Thus, suppose that $a, b \in N$. We examine the following cases:

Case 1: $a \in K$.

a) $b \notin K$.

We shall prove that there exists some positive integer n such that $a^nb = ba^n$. Thus, suppose that $a^nb \neq ba^n, \forall n \in \mathbb{N}$. Then, $a + b \neq 0, a \neq \pm 1$ and $b \neq \pm 1$. So we have

$$x = (a + b)a(a + b)^{-1}, y = (b + 1)a(b + 1)^{-1} \in N.$$

Since N is radical over K, we can find some positive integers m_x and m_y such that

$$x^{m_x} = (a + b)a^{m_x}(a + b)^{-1}, y^{m_y} = (b + 1)a^{m_y}(b + 1)^{-1} \in K.$$

Putting $m = m_xm_y$, we have

$$x^m = (a + b)a^m(a + b)^{-1}, y^m = (b + 1)a^m(b + 1)^{-1} \in K.$$

Direct calculations give the equalities

$$x^mb - y^mb + x^ma - y^m = x^m(a + b) - y^m(b + 1) = (a + b)a^m - (b + 1)a^m = a^m(a - 1),$$

from that we get the following equality

$$(x^m - y^m)b = a^m(a - 1) + y^m - x^ma.$$

If $(x^m - y^m) \neq 0$, then $b = (x^m - y^m)^{-1}[a(a^m - 1) + y^m - x^m a] \in K$, that is a contradiction to the choice of b. Therefore $(x^m - y^m) = 0$ and consequently, $a^m(a-1) = y^m(a-1)$. Since $a \neq 1, a^m = y^m = (b+1)a^m(b+1)^{-1}$ and it follows that $a^m b = ba^m$, that is a contradiction.

b) $b \in K$.

Consider an element $x \in N \setminus K$. Since $xb \notin K$, by Case 1, there exist some positive integers r, s such that

$$a^r xb = xba^r \text{ and } a^s x = xa^s.$$

From these equalities it follows that

$$a^{rs} = (xb)^{-1}a^{rs}(xb) = b^{-1}(x^{-1}a^{rs}x)b = b^{-1}a^{rs}b,$$

and consequently, $a^{rs}b = ba^{rs}$.

Case 2: $a \notin K$.

Since N is radical over K, there exists some positive integer such that $a^m \in K$. By Case 1, there exists some positive integer m such that $a^{nm}b = ba^{nm}$. ■

In [1, Theorem 5] it was shown that if D is a centrally finite division ring with the center F whose characteristic is different from the index of D over F then D^* contains no maximal subgroups that are radical over F. Now, in the case of division ring of type 2, we can prove the following theorem.

Theorem 3.2 *Let D be a division ring of type 2 with the center F such that $[D : F] = \infty$ and char$F = p > 0$. Then the group D^* contains no maximal subgroups that are radical over F.*

Proof. Suppose that M is a maximal subgroup of D^* that is radical over F. Put $G = D' \cap M$. For each $x \in G$, there exists a positive integer $n(x)$ such that $x^{n(x)} \in F$. It follows that $x^{n(x)} \in D' \cap F = Z(D')$. By Theorem 2.1, $Z(D')$ is periodic, so x is periodic. Thus, G is a periodic group. Since $M' \leq G, M'$ is a periodic too. For any $x, y \in M'$, put $H = \langle x, y\rangle$ and $D_1 = F(x, y)$. Then $n := [D_1 : F] < \infty$ and H is a periodic subgroup of $D_1^* \leq GL_n(F)$. By [8, (9.9'), p. 154], H is finite. Since *char*$F = p > 0$, by [8, (13.3), p.215], H is cyclic . In particular, x and y commute with each other, and consequently, M' is abelian. It follows that M is a solvable group. Thus M is a solvable maximal subgroup of D^*. By [1, Cor. 10] and [3, Th. 6], $[D : F] < \infty$, that is a contradiction. ■

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